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The Petrov type of a static vacuum space–time near a normal-dominated singularity

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Abstract. It is shown that, in a static vacuum space–time, the dominant term of the Riemann tensor near a normal-dominated singularity is in general of Petrov type I, although for certain values of a parameter γ that occurs in the metric it is of type D.

1. Introduction

The structure of normal-dominated singularities in static space–times has been studied by Liang (1973). Among other properties of such a singularity, it was incorrectly stated that the dominant part of the Riemann tensor near the singularity is of Petrov type N. The purpose of the present work is to show that, near the singularity, the curvature tensor is, in general, of type I, although for certain values of a parameter γ that occurs in the metric it is of type D.

In § 2 the form of the first-order metric near the singularity is derived. In § 3 the dominant terms of the components of the Riemann tensor near the singularity are evaluated and the asymptotic Petrov type of the field is determined.

2. The metric near the singularity

A normal-dominated static singularity L may be characterised intuitively as a time-like singular line boundary of a static space–time with the following properties: (i) each event p in a neighbourhood of L can be connected to L by a space-like curve of bounded arc length; (ii) if r is the normal geodesic distance of any event p from L, then the surfaces $r = \text{constant}$ are time-like 3-cylinders and, as $r \to 0$, the intrinsic curvatures of the 3-cylinders become small compared to their extrinsic curvatures. For a more detailed description of normal-dominated singularities adapted to the case of line singularities in a general space–time the reader is referred to Israel (1977).

In terms of Gaussian normal coordinates based on one of the hypersurfaces $r = \text{constant}$, the metric may be written in the form

$$\mathrm{d}s^2 = -V^2(r, x^A)\,\mathrm{d}t^2 + \mathrm{d}r^2 + g_{AB}(r, x^A)\,\mathrm{d}x^A\,\mathrm{d}x^B, \qquad A = 1, 2. \tag{2.1}$$

If

$$K_{AB} = \tfrac{1}{2} g_{AB,\,r} \tag{2.2}$$

is the extrinsic curvature of the 2-surfaces $r = \text{constant}$, $t = \text{constant}$ with respect to the hypersurface $t = \text{constant}$,

$$K^A{}_B = g^{AC}K_{CB} \qquad \text{and} \qquad K = K^A{}_A, \tag{2.3}$$

the vacuum field equations $G^i{}_j = 0$ may be written in the form

$$-2G^0{}_0 \equiv {}^{(2)}R + 2K_{,r} + K^2 + K^A{}_BK^B{}_A = 0, \tag{2.4}$$

$$G^r{}_r \equiv K_{,r} + K^A{}_BK^B{}_A + V^{-1}V_{,rr} = 0, \tag{2.5}$$

$$G^r{}_A \equiv K_{,A} - K^B{}_{A|B} + V^{-1}V_{,rA} - V^{-1}V_{,C}K^C{}_A = 0, \tag{2.6}$$

$$G^A{}_B \equiv {}^{(2)}R^A{}_B + K^A{}_{B,r} + KK^A{}_B + V^{-1}V_,{}^A{}_{|B} + V^{-1}V_{,r}K^A{}_B = 0, \tag{2.7}$$

where ${}^{(2)}R^A{}_B$ and ${}^{(2)}R$ are the Ricci tensor and scalar respectively formed from the metric g_{AB}, a comma denotes ordinary partial derivative and a vertical stroke denotes covariant derivative with respect to g_{AB}.

The requirement of normal dominance means that near the singularity the terms involving derivatives with respect to r in equations (2.4)–(2.7) are large compared with the other terms. As a first approximation to equations (2.4) and (2.7) we may therefore write

$$2K_{,r} + K^2 + K^A{}_BK^B{}_A = 0, \tag{2.8}$$

$$K^A{}_{B,r} + KK^A{}_B + V^{-1}V_{,r}K^A{}_B = 0. \tag{2.9}$$

Since, by (2.2) and (2.3),

$$K = (\ln\alpha)_{,r} \qquad \text{where } \alpha = (\det g_{AB})^{1/2}, \tag{2.10}$$

equation (2.9) takes the form

$$K^A{}_{B,r} + (\ln(\alpha V))_{,r}K^A{}_B = 0 \tag{2.11}$$

so that

$$K^A{}_B = \underset{(0)}{M}{}^A{}_B/(\alpha V), \tag{2.12}$$

where $\underset{(0)}{M}{}^A{}_B = \underset{(0)}{M}{}^A{}_B(x^A)$ are arbitrary functions of integration. Substituting (2.12) into (2.8) then yields

$$\alpha V = C_0(r - r_0), \tag{2.13}$$

where

$$C_0 = \tfrac{1}{2}(\underset{(0)}{M}{}^2 + \underset{(0)}{M}{}^A{}_B\underset{(0)}{M}{}^B{}_A)/\underset{(0)}{M}, \qquad \underset{(0)}{M} = \underset{(0)}{M}{}^A{}_A, \tag{2.14}$$

and $r_0 = r_0(x^A)$ is an arbitrary function of integration. Equation (2.12) then becomes

$$K^A{}_B = \underset{(0)}{K}{}^A{}_B/(r - r_0), \tag{2.15}$$

with $\underset{(0)}{K}{}^A{}_B = \underset{(0)}{M}{}^A{}_B/C_0$. The components of the 2-metric g_{AB} are found by solving the system of equations

$$g_{AB,r} = 2g_{AC}\underset{(0)}{K}{}^C{}_B/(r - r_0). \tag{2.16}$$

Let

$$\gamma = \underset{(0)}{M}^A{}_B \underset{(0)}{M}^B{}_A / \underset{(0)}{M}^2 \qquad (\text{so that } \infty > \gamma \geq \tfrac{1}{2}),$$

$$p_\pm = [1 \pm (2\gamma - 1)^{1/2}]/(1+\gamma), \qquad a_\pm = (p_\pm - \underset{(0)}{K}^1{}_1)/\underset{(0)}{K}^2{}_1. \tag{2.17}$$

The general solution of equation (2.16) is

$$\begin{aligned} g_{11} &= A_0(r-r_0)^{2p_+} + B_0(r-r_0)^{2p_-},\\ g_{12} &= g_{21} = A_0 a_+(r-r_0)^{2p_+} + B_0 a_-(r-r_0)^{2p_-},\\ g_{22} &= A_0 a_+^2 (r-r_0)^{2p_+} + B_0 a_-^2 (r-r_0)^{2p_-}, \end{aligned} \tag{2.18}$$

where $A_0 = A_0(x^A)$ and $B_0 = B_0(x^A)$ are arbitrary functions of integration†. A transformation $x'^A = x'^A(x^B)$ where

$$\mathrm{d}x'^1 = \phi_+(x^A)(\mathrm{d}x^1 + a_+\,\mathrm{d}x^2), \qquad \mathrm{d}x'^2 = \phi_-(x^A)(\mathrm{d}x^1 + a_-\,\mathrm{d}x^2), \tag{2.19}$$

while ϕ_+ and ϕ_- are chosen so that the right-hand sides of (2.19) are exact forms, yields a diagonal 2-metric

$$^{(2)}\mathrm{d}s^2 = g_+(x^A)(r-r_0)^{2p_+}(\mathrm{d}x^+)^2 + g_-(x^A)(r-r_0)^{2p_-}(\mathrm{d}x^-)^2 \tag{2.20}$$

where primes have been dropped, (x^1, x^2) are from now on renamed as (x^+, x^-) and $g_+(x^A)$, $g_-(x^A)$ are arbitrary functions of x^A, $A = +, -$.

Finally, from (2.13), (2.10) and (2.20), V becomes

$$V = V_0(x^A)(r-r_0)^{p_0} \tag{2.21}$$

where

$$V_0(x^A) = C_0(g_+g_-)^{-1/2}, \qquad p_0 = 1 - p_+ - p_- = (\gamma-1)/(1+\gamma), \tag{2.22}$$

and the full first-order metric is then given by

$$\mathrm{d}s^2 = -V_0^2(r-r_0)^{2p_0}\,\mathrm{d}t^2 + \mathrm{d}r^2 + g_+(r-r_0)^{2p_+}(\mathrm{d}x^+)^2 + g_-(r-r_0)^{2p_-}(\mathrm{d}x^-)^2. \tag{2.23}$$

The singularity is at $r = r_0(x^A)$. However, Liang (1973) has shown that as a result of the normal dominance conditions

$$\begin{aligned} &^{(2)}R^A{}_B(K^C{}_D K^D{}_C)^{-1} \to 0 \qquad &&\text{as } r \to r_0,\\ &V^{-1}V,^A{}_{|B}(K^C{}_D K^D{}_C)^{-1} \to 0 \qquad &&\text{as } r \to r_0, \end{aligned} \tag{2.24}$$

one may, without loss of generality, take $r = 0$ as the singularity. The metric (2.23) then becomes

$$\mathrm{d}s^2 = -V^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + g_+r^{2p_+}(\mathrm{d}x^+)^2 + g_-r^{2p_-}(\mathrm{d}x^-)^2, \tag{2.25}$$

where now

$$V = V_0(x^A)r^{2p_0}. \tag{2.26}$$

By the first two equations of (2.17), p_+ and p_- take values in the range $1 \geq p_+ \geq p_- \geq -\frac{1}{3}$, with $p_+ = 1$ when $\gamma = 1$, $p_+ = p_-$ when $\gamma = \frac{1}{2}$ and $p_- = -\frac{1}{3}$ when $\gamma = 5$. However,

† One should note here that the 2-metric components g_{AB} as given by equation (12) of Liang (1973) are incorrect and his subsequent argument for the existence of a coordinate transformation that diagonalises the metric is incomplete.

when $\gamma = 1$ the normal dominance conditions (2.24) are no longer satisfied. In what follows we shall therefore exclude this case so that we shall have

$$1 > p_+ \geqslant p_- \geqslant -\tfrac{1}{3}. \tag{2.27}$$

We note finally that the equations (2.5) and (2.6) have not been used in the derivation of the metric (2.25). Calculation of $G^r{}_r$ and $G^r{}_A$ for this metric yields

$$G^r{}_r \sim \beta(x^A)/r^2, \qquad G^r{}_A \sim \beta_{,A}(x^A)(\ln r)/r, \tag{2.28}$$

where

$$\beta(x^A) = p_+^2 + p_-^2 + p_+p_- + p_+ - p_- = 0 \qquad \text{(identically)}.$$

The symbol $\sim$ here and in what follows is taken to mean 'equals an expression of which the dominant term as $r \to 0$ is'.

3. The Petrov type

In order to determine the Petrov type of the Riemann tensor (or the Weyl tensor, since we are dealing with a vacuum field) for the metric (2.25) near the singularity, we choose a null tetrad

$$\begin{aligned} &l_a\,\mathrm{d}x^a = (-V\,\mathrm{d}t + \mathrm{d}r)/\sqrt{2}, \qquad n_a\,\mathrm{d}x^a = -(V\,\mathrm{d}t + \mathrm{d}r)/\sqrt{2}, \\ &m_a\,\mathrm{d}x^a = (g_+^{1/2} r^{p_+} \mathrm{d}x^+ + \mathrm{i} g_-^{1/2} r^{p_-} \mathrm{d}x^-)/\sqrt{2}, \\ &\bar{m}_a\,\mathrm{d}x^a = \text{complex conjugate of } m_a\,\mathrm{d}x^a. \end{aligned} \tag{3.1}$$

For the moment, let us exclude the case $\gamma = \tfrac{1}{2}$ so that we have

$$1 > p_+ > p_-. \tag{3.2}$$

The dominant terms in the null tetrad components of the Weyl tensor are given by

$$\begin{aligned} &\psi_0 \equiv -C_{abcd} l^a m^b l^c m^d \sim (p_+ - p_-)(1 - p_+ - p_-)/2r^2, \\ &\psi_1 \equiv -C_{abcd} l^a n^b l^c m^d \sim -\tfrac{1}{2} p_{-,+}(p_+ - p_-) g_+^{-1/2} \ln r / r^{1+p_+}, \\ &\psi_2 \equiv -\tfrac{1}{2} C_{abcd}(l^a n^b l^c n^d - l^a n^b m^c \bar{m}^d) \sim p_+p_-/2r^2, \\ &\psi_3 \equiv -C_{abcd} n^a l^b n^c \bar{m}^d \sim \tfrac{1}{2} p_{-,+}(p_+ - p_-) g_+^{-1/2} \ln r / r^{1+p_+}, \\ &\psi_4 \equiv -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d \sim (p_+ - p_-)(1 - p_+ - p_-)/2r^2. \end{aligned} \tag{3.3}$$

These components were calculated by means of a REDUCE computer program based on an algorithm of Campbell and Wainwright (1977). From (3.3) we see that ψ_0, ψ_2 and ψ_4 are all of order r^{-2} and dominate ψ_1 and ψ_3 as $r \to 0$.

One may summarise the procedure for determining the Petrov type of a gravitational field as follows (see, for example, d'Inverno and Russell-Clark (1971)). Let

$$I = \psi_0\psi_4 - 4\psi_1\psi_3 + 3\psi_2^2, \tag{3.4}$$

$$J = \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_1 & \psi_2 & \psi_3 \\ \psi_2 & \psi_3 & \psi_4 \end{vmatrix}, \tag{3.5}$$

$$G = \psi_0^2\psi_3 - 3\psi_0\psi_1\psi_2 + 2\psi_1^3, \tag{3.6}$$

$$H = \psi_0\psi_2 - \psi_1^2. \tag{3.7}$$

Then the Weyl tensor is
(i) at least of type II if and only if

$$I^3 = 27J^2, \tag{3.8}$$

(ii) at least of type III if and only if

$$I = J = 0, \tag{3.9}$$

(iii) of type N if and only if

$$G = H = I = J = 0, \tag{3.10}$$

(iv) of type D if and only if, in addition to (3.8),

$$G = \psi_0^2 I - 12H^2 = 0, \tag{3.11}$$

and (3.9) is not satisfied.

Substituting from (3.3) into (3.4) and (3.5) and using the second equation of (2.17), one obtains

$$I^3 - 27J^2 \sim 4(1-\gamma)^6(2\gamma-1)(\gamma-5)^2/(1+\gamma)^{12}r^{12}. \tag{3.12}$$

Hence, in general the dominant part of the asymptotic Weyl tensor is of Petrov type I. At events where $\gamma = 5$, which means that $p_+ = \frac{2}{3}$ and $p_- = -\frac{1}{3}$, the right-hand side of (3.12) vanishes so that the asymptotic Weyl tensor is at least of type II. Note that the cases $\gamma = 1$ and $\gamma = \frac{1}{2}$ have been excluded by condition (3.2). One may then verify that for $\gamma = 5$ the dominant part of the Weyl tensor satisfies (3.11) but not (3.9), so that for this case the asymptotic Weyl tensor is of type D.

At events where $\gamma = \frac{1}{2}$, which have been excluded above, one finds that $p_+ = p_- = \frac{2}{3}$ and

$$\psi_2 \sim 2/9r^2, \tag{3.13}$$

while the other ψ's are of a lower order. Hence, in this case, the asymptotic field is again of type D.

In the paper by Liang (1973) a null tetrad l'_a, n'_a, m'_a, $\bar{m}'_a$ is chosen where, in terms of the tetrad defined above,

$$l'_a = V^{-1}l_a, \qquad n'_a = Vn_a, \qquad m'_a = m_a, \tag{3.14}$$

and V is given by (2.26). The relations between the components of the Weyl tensor in this frame and those of the present work are

$$\psi'_0 = V^{-2}\psi_0, \qquad \psi'_1 = V^{-1}\psi_1, \qquad \psi'_2 = \psi_2, \qquad \psi'_3 = V\psi_3, \qquad \psi'_4 = V^2\psi_4. \tag{3.15}$$

Thus, omitting the coefficients of the powers of r and excluding the case $\gamma = \frac{1}{2}$,

$$\begin{aligned}
\psi'_0 &\sim r^{(-4+2p_+ +2p_-)} = r^{-4\gamma/(1+\gamma)},\\
\psi'_1 &\sim r^{(-2+p_-)} \ln r = r^{-(1+2\gamma+(2\gamma-1)^{1/2})/(1+\gamma)} \ln r,\\
\psi'_2 &\sim r^{-2},\\
\psi'_3 &\sim r^{-(2p_+ +p_-)} \ln r = r^{-[3+(2\gamma-1)^{1/2}]/(1+\gamma)} \ln r,\\
\psi'_4 &\sim r^{-2(p_+ +p_-)} = r^{-4/(1+\gamma)}.
\end{aligned} \tag{3.16}$$

Liang concludes that since the dominant term near the singularity is either ψ'_0 or ψ'_4, depending on whether $\gamma > 1$ or $\gamma < 1$, the asymptotic Weyl tensor is of Petrov type N. However, this conclusion is invalid. The relative dominance of the ψ's can be changed at will by a suitable scaling of l_a and n_a. On the other hand, the invariant procedure employed above is unambiguous and when applied to (3.16) leads to the results already established.

4. Conclusion

Because of the arbitrariness in the scaling of the null tetrad vectors l_a and n_a, care must be taken in determining the Petrov types of asymptotic fields. The invariant procedure of § 3 avoids any ambiguity in this regard. The application of this procedure to normal-dominated static singularities shows that the asymptotic field near the singularity is, in general, of type I except where $\gamma = 5$ and $\gamma = \frac{1}{2}$ when it is of type D.

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References

Campbell S J and Wainwright J 1977 *Gen. Rel. Grav.* **8** 987
d'Inverno R A and Russell-Clark R A 1971 *J. Math. Phys.* **12** 1258
Israel W 1977 *Phys. Rev.* D **15** 935
Liang E P T 1973 *Commun. Math. Phys.* **32** 51